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ON COMPLICATED MODELS OF CONTINUOUS MEDIA IN THE GENERAL THEORY OF RELATIVITY*

A.G. TSYPKIN

In the context of the general theory of relativity, the system of Euler's equations is obtained from the variational equation under the assumption that the Lagrangian of the material depends on supplementary (as compared with classical theories) thermodynamic parameters, and when possible irreversible processes are taken into account. It is shown that, for a thermodynamically closed system, the equations of momenta for a continuous medium are a consequence of the field equations. The form of the energy-momentum tensor of the material is considered when the arguments include the Lagrangian of the derivatives of the supplementary thermodynamic parameters.

Let x^i be the coordinates in four-dimensional Riemann space, in which the components of the metric tensor g_{ij}, the coefficients of parallel transfer $\Gamma_{ij}{}^k$, and the curvature tensor $R_{ijk}^{\cdot\cdot\cdot s}$ are connected by the equations

$$
\begin{aligned}
& ds^2 = g_{ij} dx^i dx^j \\
& \Gamma_{ij}{}^k = \frac{1}{2} g^{ks} \left[\frac{\partial g_{ks}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s} \right] \\
& R_{ijk\cdot}^{\cdot\cdot\cdot s} = \frac{\partial \Gamma_{ik}^s}{\partial x^j} - \frac{\partial \Gamma_{jk}^s}{\partial x^i} + \Gamma_{pj}^s \Gamma_{ki}^p - \Gamma_{pi}^s \Gamma_{kj}^p \\
& R_{ij} = R_{isj\cdot}^{\cdot\cdot s}, \quad R = R_{ij} g^{ij}
\end{aligned}
\tag{1}
$$

where R_{ij} are the components of the Ricci tensor, and R is the scalar curvature of the space (the Ricci scalar). Throughout, the small Latin indices cover the values 1, 2, 3, 4; summation is performed with respect to repeated sub- and super-scripts; the signature of the metric is $(+ - - -)$.

Together with the variables x^i in the Riemann space we consider for a solution the accompanying coordinates ξ^k, in which fixed values ξ^1, ξ^2, ξ^3 individualize a point of the continuous medium; we assume that there is a one-to-one correspondence $x^i = x^i(\xi^k)$ between the variables x^i and ξ^k, which is the law of motion of the continuum of the continuous medium.

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In the contex of the general theory of relativity, the dynamic equations for the field and continuous medium can be obtained by means of the variational equation /1/

$$\delta \int_{V_4} \Lambda\, dV_4 + \delta W^* + \delta W = 0 \tag{2}$$
$$dV_4 = \sqrt{-g}\, dx^1 dx^2 dx^3 dx^4,\ g = \det \| g_{ij} \|$$

where dV_4 is the invariant element of a four-dimensional volume V_4, bounded by the three-dimensional surface Σ_3, and the Lagrangian Λ, which is a four-dimensional scalar, is a function of the following arguments:

$$x_j^{\ i} \equiv \frac{\partial x^i}{\partial \xi^j},\ \mu^A,\ \nabla_k \mu^A,\ \Gamma_{ij}^{\ k},\ \frac{\partial \Gamma_{ij}^{\ k}}{\partial x^s},\ K^{\wedge B}, K^C \tag{3}$$

where the functions μ^A can be either scalars or the components of vectors or tensors of any rank with any structure of the indices, dependent on the coordinates x^k. Here, for simplicity, we do not include among the arguments covariant derivatives of higher than the first order of the arguments μ^A, or covariant derivatives of arguments $x_j^{\ i}$. Given suitable modifications of the expressions, all our deductions below remain valid for the extended set of arguments.

As the parameters μ^A we can choose the tensor characteristics of the electromagnetic field or the kinematic and thermodynamic characteristics of the continuous medium, e.g., the deformation tensor, the velocity vector, the entropy, or the temperature; ∇_k denotes the covariant derivative in the x^k coordinate system; $K^{\wedge B}(\xi^k)$, $K^C(x^k)$ are the components of the parametrically specified vectors or tensors in the accompanying ξ^k coordinate system and in the x^i coordinate system respectively, which characterize the given properties of the material and space. Among the tensors $K^{\wedge B}$ are included e.g., the given physical characteristics of the medium (e.g., the components of the tensors which characterize the anisotropic properties, the permittivity, or the permeability, etc.). Among the tensors K^C, depending on the model employed, are included the material characteristics, which are assumed to be given when constructing the model of the continuum and when making hypotheses about the geometry of the space. For instance, as the parametrically known functions $K^C(x^k)$, we can choose the characteristics of the unperturbed electromagnetic field given in the statement of the problem, or of the given unperturbed gravitational field, when we take the model of a material for which the energy depends on the field characteristics.

In the context of the models considered below, the arguments μ^A are known functions, as are the components of the metric tensor $g_{ij}(x^k)$ and the law of motion of the continuous medium $x^i(\xi^k)$.

Consider the fairly general model in which the Lagrangian Λ is taken in the form

$$\Lambda = -R/(2\varkappa) + \nabla_i \Omega^i + \Lambda_m,\ \varkappa = 8\pi G/c^4 \tag{4}$$

where $\varkappa$ is the gravitational constant, G is the Newtonian gravitational constant, c is the velocity of light in vacuo; the four-dimensional scalar Λ_m is the material Lagrangian, which depends on the set of arguments (3) (here, by "material" we mean a set of continuous media and physical fields, excluding the gravitational field); Ω^i are the components of the four-dimensional vector, which is given in the context of the particular model and is a function of the coordinates x^i only.

Our future expressions can be extended to the case when the vector components Ω^i also depend on the set of arguments (3). The physical meaning of Ω^i was discussed in detail in /2/. Below, we assume for simplicity that the function Λ_m, which characterizes the thermodynamic state of the material and depends on the set of arguments (3), does not depend on $\partial \Gamma_{ij}^{\ k}/\partial x^s$ and depends on $\Gamma_{ij}^{\ k}$ only via the argument $\nabla_k \mu^A$. (A similar set of arguments, in which μ^A were regarded as vector components, and instead of $\Gamma_{ij}^{\ k}$ the argument $\partial g_{ij}/\partial x^s$ was taken, was used in /3/).

We shall henceforth consider the simple case of continuous motions and processes, when, in general, the non-holonomic functional δW^* of the variational equation has the form

$$\delta W^* = \int_{V_4} \left(M_A \delta_L \mu^A - \frac{1}{2} \tau^{ij} \delta_L g_{ij} \right) dV_4 \tag{5}$$
$$\delta_L \mu^A = \partial \mu^A + \delta x^i \nabla_i \mu^A - F_{Bi}^{Ak} \mu^B \nabla_k \delta x^i$$

Here, M_A are the components of the generalized forces, whose specification is connected with the theory of internal dissipative mechanisms in the thermodynamic system used in the model, $\tau^{ij} = \tau^{ji}$ are the components of the tensor defining the viscous properties of the medium, and $\delta_L \mu^A$ is the absolute variation of the tensor μ^A components, which, for actual motions and processes, is the increment of the tensor μ^A components with respect to the accompanying coordinate system; F_{Bi}^{Ak} is a combination of Kronecker deltas, whose form can be established from the relation

$$\nabla_i \mu^A = \partial\mu^A/\partial x^i + F_{Bs}^{Ak}\mu^B\Gamma_{ik}^s$$

while the variations $\delta_L g_{ij}$ are given by the equations

$$\delta_L g_{ij} = \partial g_{ij} + g_{sj}\nabla_i \delta x^s + g_{is}\nabla_j \delta x^s$$

If the thermodynamic system is closed, i.e., there is no force or energy interaction with other thermodynamic systems, then, by the second law of thermodynamics for actual processes, we have the equation

$$M_A d_L \mu^A - (1/2)\tau^{ij} d_L g_{ij} = 0$$

The presence in the variational equation of a non-zero non-holonomic functional δW^* is in general essential when constructing many models of continuous media and fields. Here, the set of varied parameters μ^A in the expression for the functional δW^* may obviously not be the same as the set of parameters μ^A which appear among the arguments of the Lagrangian (3). It must also be noted that the dependence of the non-holonomic functional δW^* on the absolute variations of the parameters μ^A is entirely natural, since, in accordance with its physical meaning, the parameters μ^A describe the thermodynamic state of the material, which does not depend on the choice of coordinates x^i. In particular, as will be shown below, given a suitable choice of parameters μ^A and of the generalized mass and surface forces M_A, the functional (5) enables account to be taken of the processes of dissipative heat liberation in the continuous medium, due to its electrically conducting properties.

Using the expressions for the variations, /4, 5/, the first term of the variational Eq. (2) can be written as

$$\begin{aligned}
\delta\int_{V_4}\Lambda\, dV_4 = \int_{V_4}\Big\{\Big[\Big(\frac{1}{2\varkappa}R^{ij} - \frac{1}{2}\cdot\frac{R}{2\varkappa}g^{ij} + {} \\
\frac{1}{2}\Lambda_m g^{ij} + \frac{\partial\Lambda_m}{\partial g_{ij}} + \frac{1}{2}\nabla_s\Big[\frac{\partial\Lambda_m}{\partial\nabla_l\mu^A}\Big(\frac{\partial\nabla_l\mu^A}{\partial\Gamma_{ij}^k}g^{sk} - {} \\
\frac{\partial\nabla_l\mu^A}{\partial\Gamma_{sj}^k}g^{ik} - \frac{\partial\nabla_l\mu^A}{\partial\Gamma_{si}^k}g^{jk}\Big)\Big]\Big]\partial g_{ij} + \Big[-\frac{\partial\Lambda_m}{\partial x_j^s}\nabla_i x_j^s - {} \\
\nabla_s\Big(\frac{\partial\Lambda_m}{\partial x_j^i}x_j^s\Big)\Big]\delta x^i + \Big[\frac{\partial\Lambda_m}{\partial\mu^A} - \nabla_s\Big(\frac{\partial\Lambda_m}{\partial\nabla_s\mu^A}\Big)\Big]\partial\mu^A + {} \\
\frac{\partial\Lambda_m}{\partial K^{\wedge B}}\partial K^{\wedge B} + \frac{\partial\Lambda_m}{\partial K^C}\partial K^C\Big\}dV_4 + {} \\
\int_{\Sigma_3}\Big\{\Big[-\frac{1}{2}\frac{\partial\Lambda_m}{\partial\nabla_l\mu^A}\Big(\frac{\partial\nabla_l\mu^A}{\partial\Gamma_{ij}^s}g^{sk} - \frac{\partial\nabla_l\mu^A}{\partial\Gamma_{kj}^s}g^{is} - {} \\
\frac{\partial\nabla_l\mu^A}{\partial\Gamma_{ki}^s}g^{js}\Big)\Big]\partial g_{ij} + \Big[-\frac{R}{2\varkappa}\delta_i^k + \nabla_s\Omega^s\delta_i^k - \nabla_i\Omega^k + {} \\
\frac{\partial\Lambda_m}{\partial x_j^i}x_j^k + \Lambda_m\delta_i^k - \frac{\partial\Lambda_m}{\partial\nabla_k\mu^A}\nabla_i\mu^A + \frac{\partial\Lambda_m}{\partial K^C}F_{Ai}^{Ck}K^A\Big]\delta x^i + {} \\
\frac{\partial\Lambda_m}{\partial\nabla_k\mu^A}\delta\mu^A - \Big[\frac{1}{2\varkappa}(g^{si}g^{kj} - g^{ij}g^{sk})\Big]\nabla_s\partial g_{ij}\Big\}n_k\, d\sigma_3
\end{aligned} \tag{6}$$

Here, n_k are the components of the unit outward normal vector to the three-dimensional surface Σ_3, and $d\sigma_3$ is the three-dimensional invariant element of the surface Σ_3.

From the condition for the volume integral in variational Eq. (2) to vanish, using (5) and (6) with linearly independent variations ∂g_{ij}, δx^i, and $\partial\mu^A$, we obtain the following system of Euler equations for the functions $g_{ij}(x^k)$, $x^i(\xi^k)$, $\mu^A(x^k)$:

$$\frac{1}{\varkappa}R^{ij} - \frac{1}{2\varkappa}Rg^{ij} = T^{ij} \tag{7}$$

$$\begin{aligned}
-\frac{\partial\Lambda_m}{\partial x_j^s}\nabla_i x_j^s - \nabla_s\Big(\frac{\partial\Lambda_m}{\partial x_j^i}x_j^s\Big) + M_A\nabla_i\mu^A + {} \\
\nabla_k\tau_{i\cdot}^{\cdot k} + \nabla_k(M_A F_{Bi}^{Ak}\mu^B) - \frac{\partial\Lambda_m}{\partial K^{\wedge B}}\nabla_i K^{\wedge B} - {} \\
\frac{\partial\Lambda_m}{\partial K^C}\nabla_i K^C - \nabla_s\Big(\frac{\partial\Lambda_m}{\partial K^C}F_{Bi}^{Cs}K^B\Big) = 0
\end{aligned} \tag{8}$$

$$\frac{\partial\Lambda_m}{\partial\mu^A} - \nabla_s\Big(\frac{\partial\Lambda_m}{\partial\nabla_s\mu^A}\Big) + M_A = 0 \tag{9}$$

In Eqs. (7), T^{ij} are the components of the symmetric tensor given by

$$T^{ij} = -\frac{2}{\sqrt{-g}}\frac{\partial(\Lambda_m\sqrt{-g})}{\partial g_{ij}} - \frac{1}{\sqrt{-g}}\nabla_s\left\{\frac{\partial(\Lambda_m\sqrt{-g})}{\partial\nabla_l\mu^A}\times\right.$$
$$\left.[{}^1/_2(F^{Ai}_{Bk}\delta_l^j + F^{Aj}_{Bk}\delta_l^i)\mu^B g^{ks} - F^{As}_{Bk}\mu^B(g^{kj}\delta_l^i + g^{ki}\delta_l^j)]\right\} + \tau^{ij}$$

Eqs. (7) define the metric tensor of four-dimensional Riemann space; (8) are the four-dimensional equations of momenta for the material; (9) can be regarded as the equations of state, among which, in particular, given a suitable choice of the parameters μ^A, are the equations of moments for the material.

It must be said that the equations of momenta (8) are obtained under certain special conditions on the form of the variations of the parametrically specified tensors $K^{\wedge B}(\xi^k)$, $K^C(x^k)$, namely: since, in accordance with our above assumption, the tensors $\mathbf{K}(\xi^k)$ are regarded as given, the total variations of the components of these tensors are zero

$$\delta K^{\wedge B} = 0 \tag{10}$$

and the partial variations $\partial K^{\wedge B}$ are given by

$$\partial K^{\wedge B} = -\delta x^i\nabla_i K^{\wedge B}$$

for actual motions, the equation $\delta K^{\wedge B} = 0$ in the accompanying ξ^k coordinate system is the condition that the components of the tensor $K^{\wedge B}$ be constant for an individual point of the continuous medium: $dK^{\wedge B}/ds = 0$, which does not exclude possible spatial inhomogeneity of the medium, characterized by the tensor $\mathbf{K}(\xi^k)$.

From Eqs. (10) we have the expressions for the total variations of the tensor K components in the x^i coordinate system:

$$\delta K^C = F^{Ck}_{Ai}K^A\nabla_k\delta x^i$$

while for the partial variations of the tensor K^C components we have

$$\partial K^C = -\delta x^i\nabla_i K^C + F^{Ck}_{Ai}K^A\nabla_k\delta x^i$$

It can be shown that the equations of momenta (8) retain their form when the arguments include only the parametrically specified tensors with components with respect to the accompanying coordinate system $K^{\wedge B}(\xi^k) = \xi_A{}^B K^A(x^k)$, where $\xi_A{}^B$ are products of the type $\xi_j{}^i\,\xi_n{}^m\ldots$ ($\|\xi_j{}^i\|$ is the inverse of the matrix $\|x_j{}^i\|$). We only need to take account of the difference in the dependence of the function Λ_m on the argument $x_j{}^i$.

From the condition for the surface integral in variational Eq. (2) to vanish we obtain

$$\delta W = \int_{\Sigma_3}(P_{i\cdot}^{\cdot k}\delta x^i + T^{kij}\partial g_{ij} + T^{ksij}\nabla_s\partial g_{ij} + M_A{}^k\delta\mu^A)\,n_k\,d\sigma_3 \tag{11}$$

where the quantities $P_{i\cdot}^{\cdot k}$, T^{kij}, T^{ksij}, $M_A{}^k$ are given by equations which can be found from (5) and (6). In particular, given any variations δx^i and $\delta\mu^A$ on the surface Σ_3, it follows from (5) and (6) that

$$P_{i\cdot}^{\cdot k} = \frac{1}{2\varkappa}R\delta_i{}^k - \nabla_s\Omega^s\delta_i{}^k + \nabla_i\Omega^k + \theta_{i\cdot}^{\cdot k} \tag{12}$$

Here,

$$\theta_{i\cdot}^{\cdot k} = -\frac{\partial\Lambda_m}{\partial x_j{}^i}x_j{}^k - \Lambda_m\delta_i{}^k + \frac{\partial\Lambda_m}{\partial\nabla_k\mu^A}\nabla_i\mu^A + \tag{13}$$
$$M_A F^{Ak}_{Bi}\mu^B - \frac{\partial\Lambda_m}{\partial K^A}F^{Ak}_{Bi}K^B$$

The equations that define the integrals in δW in the well-known classical theories (the theory of an ideal liquid, the theory of elasticity, etc.) are equations of state. Since the Lagrangian Λ_m is a four-dimensional scalar, the set of $P_{i\cdot}^{\cdot k}$ is a set of components of a second-rank tensor. The expression for δW given by (11) does not enable the expressions for the components of the tensors $P_{i\cdot}^{\cdot k}$, T^{kij}, T^{ksij} to be uniquely defined. To define the components of $P_{i\cdot}^{\cdot k}$ uniquely we need extra assumptions, about which variations of the parameters μ^A are regarded as independent, and these assumptions are connected with the physical interpretation of the equations of state. The form of the tensor functions T^{kij} and T^{ksij} likewise cannot be uniquely established, since the quantities ∂g_{ij} and $\nabla_s\partial g_{ij}$ are not independent on the surface Σ_3. It was shown in /6/ that, from the expression for δW, there follow only expressions for certain definite combinations of components of the tensors T^{kij} and T^{ksij}, resulting from the conditions for the variations ∂g_{ij} and $(D/Dn)(\partial g_{ij}) \equiv n^k\nabla_k\partial g_{ij}$

on the surface Σ_3 to be independent.

As a special case of the general model of a continuum we take the model of a polarized and magnetized medium in which the dissipative processes are due solely to the liberation of Joule heat. Assuming that the parameter μ^A is the component of a four-dimensional vector potential of the electromagnetic field A_k, and that the generalized forces M_A are the components of the four-dimensional vector of the electric current I^k, the functional δW^* has the form

$$\delta W^* = \int_{V_4} I^k \delta_L A_k \, dV_4$$

and defines the possible liberation of Joule heat in the medium /7/.

We put the Lagrangian Λ_m equal to /7/

$$\Lambda_m = -U - L + \frac{1}{8\pi} F_{mn} H^{mn} - \frac{1}{4\pi} H^{mn} \nabla_m A_n$$

where U is the total energy density of the medium, L is a term which takes account of the interaction of the electromagnetic field with the medium, and F_{mn} and H^{mn} are the components of the electromagnetic field tensors. We shall henceforth assume that U depends on x_j^i, g_{ij}, the entropy $S, K^{\wedge B}$, and K^C, while L depends on F_{ij} and the covariant derivatives $\nabla_k F_{ij}$ as well as on the arguments listed.

In the present case, the expression for δW takes the form

$$\delta W = \int_{\Sigma_3} \left[\theta_{i\cdot}^{\cdot k} \delta x^i + \frac{1}{4\pi} H^{ks} \delta_L A_s \right] n_k \, d\sigma_3$$

where the tensor

$$\theta_{i\cdot}^{\cdot k} = \frac{\partial (U + L)}{\partial x_j^i} x_j^k - U \delta_i^k + S_{i\cdot}^{\cdot k}$$

can be interpreted as the total energy-momentum tensor of the medium + electromagnetic field system, and $S_{i\cdot}^{\cdot k}$ are the components of the Minkowski energy-momentum tensor of the electromagnetic field. (In the expressions of the present example we quote only the terms which are obtained with variation of the arguments x^i_j and A_k of the function Λ_m. The complete system of equations of mechanics and electrodynamics is similarly obtained in /7/).

When obtaining the system of Euler's equations and the expression for the functional δW, we have not used the condition that the material Lagrangian Λ_m be scalar, which enables a connection to be found between the components of the tensors T^{ij} and θ^{ij}. From the condition for Λ_m to be scalar under any infinitesimal coordinate transformation $y^i = x^i + \delta\eta^i(x^k)$ there follow the equations

$$\theta^{ij} = T^{ij} + \nabla_s \left[\frac{\partial \Lambda_m}{\partial \nabla_i \mu^A} (F_{Bk}^{Aj} g^{ks} - F_{Bk}^{As} g^{kj}) \mu^B \right] = \tag{14}$$

$$-\frac{2}{\sqrt{-g}} \frac{\partial (\Lambda_m \sqrt{-g})}{\partial g_{ij}} + \nabla_s \left[\frac{\partial \Lambda_m}{\partial \nabla_s \mu^A} F_{Bk}^{Aj} \mu^B g^{ik} \right]$$

which show that, when the set (3) of defining parameters include the arguments $\nabla_s \mu^A$, where μ^A are the components of a vector or tensor of any rank, the components of the tensor θ^{ij} are asymmetric. Obviously, when $\nabla_s \mu^A$ are not present in the set (3), we see from (14) that the tensors θ^{ij} and T^{ij} are identical.

Consider some consequences of the system of Euler's Eqs.(7)-(9) and identities (14). From the gravitational field Eqs.(7) in the light of Bianchi's identities we obtain the equations

$$\nabla_j T^{ij} = 0 \tag{15}$$

In view of (13), the equations of momenta (8) can be given the form

$$\nabla_k \theta_{i\cdot}^{\cdot k} = \frac{\partial \Lambda_m}{\partial \nabla_s \mu^A} (\nabla_s \nabla_i - \nabla_i \nabla_s) \mu^A$$

or

$$\nabla_k \theta_{i\cdot}^{\cdot k} = R_{likx} \frac{\partial \Lambda_m}{\partial \nabla_j \mu^A} F_{Bj}^{As} \mu^B g^{jk} \tag{16}$$

Using the connection (14) between the tensors θ^{ij} and T^{ij}, we can write Eqs.(15) as

$$\nabla_k \theta^{ik} = (1/2)(\nabla_s \nabla_k - \nabla_k \nabla_s) \theta^{isk} \tag{17}$$

$$\theta^{isk} = -\theta^{iks} = \frac{\partial \Lambda_m}{\partial \nabla_i \mu^A} (F^{As}_{Bj} g^{jk} - F^{Ak}_{Bj} g^{js}) \mu^B$$

It can be shown that Eqs.(16) and (17) are the same, i.e., in the context of the present model Eqs.(8) of the momenta of the material are a consequence of Eqs.(7). This property of the system of Euler's equations is destroyed if there are terms in δW^* that describe external spatial interaction of the thermodynamic system with other such systems, but is retained regardless of the method of specifying the generalized interior surface and mass forces τ^{ij} and M_A, corresponding to internal mechanisms of transformation of one type of energy into another within the thermodynamic system, or of the type of parametrically specified tensors K^{AB} and K^C.

In the above examples of energy dissipation due to the viscous and electrically conducting properties of the continuous medium and the transformation of non-thermal into thermal energy, the equations of material momenta are a consequence of the field equations, regardless of the chosen type of dependence of the components of the viscous stress tensor τ^{ij} and of the four-dimensional current vector I^k on the defining parameters (3). It must also be noted that, when account is taken of dissipative effects due to the electrical conduction of the medium, we have to include in the material Lagrangian the electromagnetic field Lagrangian, since otherwise the Joule heat liberation cannot be regarded as an internal process of one type of energy transformation into another within a single thermodynamic system.

The above system of Euler's Eqs.(7)-(9) and the expression for the functional δW show that, when constructing complicated models of continuous media in the general theory of relativity, the question arises of what to call the material energy-momentum tensor. The equations of momenta for the medium can be written in the form (15) with the aid of the tensor T^{ij}, which is usually called the material energy-momentum tensor. Given any material continuum, this tensor is symmetric regardless of the choice of model defining parameters. However, it can be seen from the variational equation that the conditions for strong discontinuities on the surface (and also the initial and boundary conditions) are stated for the components, given by (14), of the tensor P^{ij}, which in general is not symmetric and differs from T^{ij} /8/. The difference is connected, first with the fact that the expression for the Lagrangian Λ_m may include the divergence term $\nabla_s \Omega^s$, which does not affect the form of Euler's Eqs.(7) or of the tensor T^{ij}, but changes the form of P^{ij} (the form of the vector Ω^s, and its possible physical interpretation, are discussed in detail in /9/), and second, with the presence in the set of arguments (3) of the covariant derivatives $\nabla_i \mu^A$. Everything said above refers to models of continua with complicated physico-chemical properties. If the covariant derivatives of the tensor functions μ^A and the term $\nabla_s \Omega^s$, are not present in the arguments of the Lagrangian Λ_m, the components of the tensor P^{ij} are the same, apart from a factor, as the components of the Ricci tensor.

When constructing models of a continuum in the context of the special theory of relativity, with the metric space postulated as a four-dimensional pseudo-Euclidean Minkowski space, the equations of the momenta for the material (16) can be written in the form $\nabla_k \theta_{i\cdot}^{\cdot k} = 0$, as well as in the form $\nabla_k T_{i\cdot}^{\cdot k} = 0$, where the components of the asymmetric tensor θ^{ik} and of the symmetric tensor T^{ik} are connected by Eqs.(14), while the boundary conditions are stated for the components of the asymmetric tensor $\theta^{ik} \equiv P^{ik}$.

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